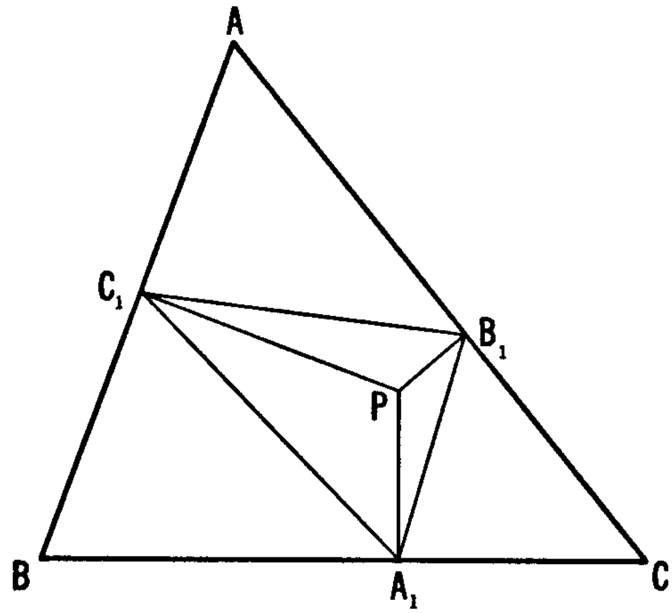

Figure 1.9A

Let us examine Figure 1.9A more closely. The right angles at B_1 and C_1 indicate that these points lie on the circle with diameter AP; in other words, P lies on the circumcircle of $\triangle AB_1C_1$. Applying the Law of Sines to this triangle and also to $\triangle ABC$ itself, we obtain

$$\frac{B_1C_1}{\sin A} = AP, \qquad \frac{a}{\sin A} = 2R,$$

whence

$$B_1C_1 = a\frac{AP}{2R}.$$

Similarly,

$$C_1A_1 = b\frac{BP}{2R} \quad \text{and} \quad A_1B_1 = c\frac{CP}{2R}.$$

We have thus proved:

THEOREM 1.91. *If the pedal point is distant* x, y, z *from the vertices of* $\triangle ABC$, *the pedal triangle has sides*

$$\frac{ax}{2R}, \qquad \frac{by}{2R}, \qquad \frac{cz}{2R}.$$

The case when $x = y = z = R$ is, of course, familiar.

An interesting exercise involving pedal triangles of pedal triangles is at the same time a delightful example of imagination in geometry. It seems to have first appeared when it was added, by the editor J. Neuberg, to the sixth edition (1892) of John Casey's classic *A Sequel to the First Six Books of the Elements of Euclid.* In Figure 1.9B an interior point P has been used to determine $\triangle A_1B_1C_1$, the (first) pedal triangle of $\triangle ABC$. The same pedal point P has been used again to determine $\triangle A_2B_2C_2$, the pedal triangle of $\triangle A_1B_1C_1$, which we naturally call the "second pedal triangle" of $\triangle ABC$. A third operation yields $\triangle A_3B_3C_3$, the pedal triangle of $\triangle A_2B_2C_2$. The understanding is that, for this "third pedal triangle" also, we use the same pedal point P. In this terminology, Neuberg's discovery can be expressed thus:

THEOREM 1.92. *The third pedal triangle is similar to the original triangle.*

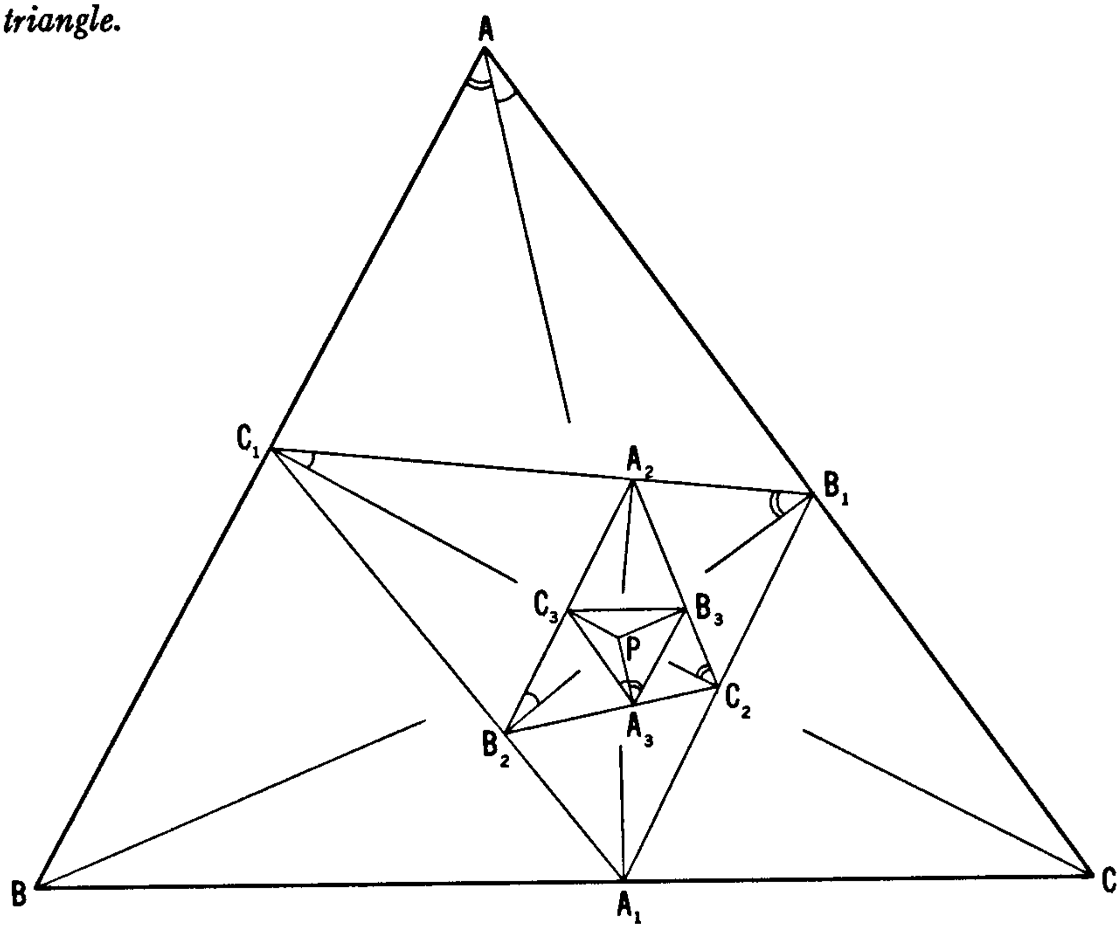

Figure 1.9B

The proof is surprisingly simple. The diagram practically gives it away, as soon as we have joined P to A. Since P lies on the circumcircles of all the triangles AB_1C_1, $A_2B_1C_2$, $A_3B_3C_2$, $A_2B_2C_1$, and $A_3B_2C_3$, we have

$$\angle C_1AP = \angle C_1B_1P = \angle A_2B_1P = \angle A_2C_2P$$
$$= \angle B_3C_2P = \angle B_3A_3P$$

and

$$\angle PAB_1 = \angle PC_1B_1 = \angle PC_1A_2 = \angle PB_2A_2$$
$$= \angle PB_2C_3 = \angle PA_3C_3.$$

In other words, the two parts into which AP divides $\angle A$ (marked in the diagram with a single arc and a double arc) have their equal counterparts at B_1 and C_1, again at C_2 and B_2, and finally both at A_3. Hence ΔABC and $\Delta A_3B_3C_3$ have equal angles at A and A_3. Similarly, they have equal angles at B and B_3. Thus the theorem is proved.

It is interesting to follow in the diagram the "parade of angles" from position A to position A_3: as neat as the maneuvers of a drill team.

This property of continued pedals has been generalized by B. M. Stewart (*Am. Math. Monthly*, vol. 47, Aug.-Sept. 1940, pp. 462-466). He finds that *the nth pedal n-gon of any n-gon is similar to the original n-gon.* It is instructive to try this for the fourth pedal quadrilateral of a quadrilateral.

At this point let us pause in our investigations. We have done part of what we set out to do: beginning with well-known data, we have developed a few simple but significant facts. There are many problems that lend themselves to solution by the methods described here. Some of them are well-known posers that the reader may have seen before. We bring this chapter to a close by presenting five of these hardy perennials.

EXERCISES

1. If a cevian AQ of an equilateral triangle ABC is extended to meet the circumcircle at P, then

$$\frac{1}{PB} + \frac{1}{PC} = \frac{1}{PQ}.$$

2. If an isosceles triangle PAB, with equal angles $15°$ at the ends of its base AB, is drawn inside a square $ABCD$, as in Figure 1.9C, then the points P, C, D are the vertices of an equilateral triangle.

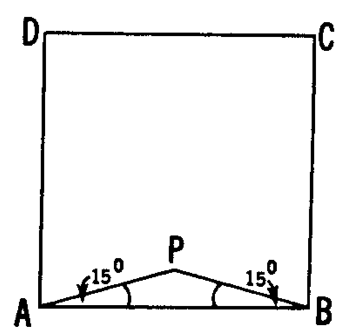

Figure 1.9C

3. If lines PB and PD, outside a parallelogram $ABCD$, make equal angles with the sides BC and DC, respectively, as in Figure 1.9D, then $\angle CPB = \angle DPA$. (Of course, this is a plane figure, not three dimensional!)